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A NEW THEORY OF THE RULE OF SIGNS.

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(*Continued from page* 88.)

For all equations of the third degree, the cubic type is decisive; if it does not indicate imaginary roots, there are none; as appears from comparison with Sturm's theorem. For the fourth and higher degrees, it is decisive only so far as it indicates two imaginary or equal roots; leaving the question of more of them, yet to be decided.

Example. $x^3 + 2x^2 - 23x - 70 = 0.$

$f' = -3 \times 23 \div 2^2 = -17.25,\ g' = -3 \times 2 \times 70 \div 23^2 = -.794.$

Since g' and f' are interchangeable, we enter the foregoing Table with -0.8 nearest value of g', and there find that f' must negatively exceed -14.42, which it does, being -17.25, and indicating two imaginary roots.

Example. $x^3 - 5x^2 + 2x + 12 = 0.$

$$f' = +6 \div 5^2 = .24,\ g' = -180 \div 2^2 = -45.$$

Entering the table with $f' = 0.20$ and interpolating for .04, we find for $f' = 0.24$, $g_1 = -52.56$. As g' is *within* this limit, the roots are *all real*, and accord with the Rule of signs. The result is confirmed by computing the criterion (C) with the respective values of f' and g'; and the comparative smallness of (C), being $+0.6$, indicates two roots to be nearly equal.

VII. *The Biquadratic Type*, or five consecutive terms:

$$x^m + \ldots . [Px^n + Qx^{n-1} + Rx^{n-2} + Sx^{n-3} + Tx^{n-4}] + \ldots . = 0.$$

Differentiating $n - 4$ times and proceeding as with former types, also making $m - n = m'$, the final derivative takes the form of

$$x^4 + \frac{4(m'+1)}{n}Q'x^3 + \frac{6(m'+1)(m'+2)}{n(n-1)}R'x^2 + \frac{4(m'+1)\ldots(m'+3)}{n\,.\,.\,(n-2)}S'x$$
$$+ \frac{(m'+1)\ldots(m'+4)}{n\ldots(n-3)}T' = 0.$$

Or making $x = \frac{m'+1}{n}\left(z - Q'\right)$, and reducing,

$$z^4 - 6Q'^2(1-f')z^2 + 8Q'^3(1 - \tfrac{3}{2}f' + \tfrac{1}{2}f'^2g')z + Q'^4(-3 + 6f' - 4f'^2g' + f'^3g'^2h') = 0.$$

Here $Q' = Q \div P$; $h = \frac{n-2}{n-3} \cdot \frac{m'+4}{m'+3}$, and $h' = \frac{hRT}{S^2}$.

For convenience, we may name the similar quantities, f', g', h',.... *triple ratios.*

A decisive criterion is now given by Descartes' or Euler's well known solution of the equation $z^4 + bz^2 + cz + d = 0$, by means of the auxiliary cubic, $y^3 + 2by^2 + (b^2 - 4d)y - c^2 = 0$. This cubic, by substituting $y = y' - \frac{2}{3}b$, becomes, $y'^3 - (\frac{1}{3}b + 4d)y' - \frac{2}{27}b^3 + \frac{8}{3}bd - c^2 = 0$.

The well known criterion of $y'^3 + qy' + r = 0$, being $-(\frac{1}{3}q)^3 - (\frac{1}{2}r)^2 =$ a negative quantity, we substitute from the above equation of z^4 and find the criterion of two imaginary roots in the cubic of Euler's solution to be, after reduction,

$$(1 - \tfrac{4}{3}g' + \tfrac{1}{3}f'g'^2h')^3 - [1 - 2g' + g'^2(f' + h' - f'h')]^2 = neg. \quad \text{Or,}$$

$$(R^2 - \tfrac{4}{3}gQS + \tfrac{1}{3}fg^2hPT)^3 - [(R^2 - gQS)^2 - g^2(Q^2 - fPR)(S^2 - hRT)]^2.R^2 = negative.$$

When the middle coefficient R is zero, the criterion is simply,

$$\tfrac{1}{4}fghPT - QS = neg.$$

And when the given equation is a biquadratic, $\frac{1}{4}fgh$ is 4.

When the second coefficient Q is zero, we can at once apply the cubic type to the auxiliary cubic of Euler's solution above.

$$\text{Thus,} \qquad f' = \frac{3}{4}\left(\frac{b^2 - 4d}{b^2}\right) = \frac{3}{4} - \frac{3d}{b^2};\ g' = -\frac{c^2}{(\frac{2}{3}b)^3 f'^2}.$$

And the foregoing Table fo Limits will now determine the presence or the absence of two imaginary roots in the cubic. As shown in Algebra, the roots of the biquadratic will be all real, all imaginary, or only two imaginary, according as the cubic equation has three positive, two negative, or two imaginary roots, respectively.

VIII. *General Principle and Rule of Signs.* From the nature of the preceding demonstrations, it is evident that they are but particular cases under the following general principle:

The consecutive portions of a given equation, including contiguous terms, may form minor equations to be placed equal to zero; and each or its derivative, resolved so far as to find the signs or nature of its roots. The total number of imaginary roots so found, will be an inferior limit, and the total number of positive and of negative real roots will be superior limits to the respective numbers of roots in the original equation.

Example. $x^7 - 2x^5 + 2x^3 - 2x^2 + 2x - 1 = 0$.

Assume $x_1{}^7 - 2x_1{}^5 + 2x_1{}^3 = 0$, $2x_2{}^3 - 2x_2{}^2 = 0$, $-2x_3{}^2 + 2x_3 - 1 = 0$.

$$x_1 = \pm\sqrt{[1 \pm \sqrt{(-1)}]},\ x_2 = +\ ,\ x_3 = \tfrac{1}{2} \pm \sqrt{(-\tfrac{1}{4})}.$$

Thus of the seven roots, six must be imaginary, and one positive. The general precept evidently admits the application of all compatible rules of solution or of previous differentiation. But in general practice, it will be most convenient to suppose the equation resolved first into binomials then into trinomials, cubic types, and so on, and the completed results given by inspection according to the following

General Rule. Where the decrement is an odd number as 1, 3, *or* 5, *from one exponent to the next, if there be* NO *change from the former sign to the latter, write the* NEGATIVE *sign; but otherwise, the positive sign, in the scale of roots.*

Where the decrement is an even number, as 2, 4, 6, *from one exponent to the next, compare the two signs, and if there be* NO *change make* NO *entry, but otherwise enter both the unlike signs* + — *in the scale of roots.*

The number of positive roots in the scale, descending by 2, *if possible, will be the alternative number of real positive roots in the given equation. And so with the number of negative roots. An odd number here proves the certain existence of one root of the species.*

Note 1. The same scale of signs generally exhibits the detailed order of magnitude of the roots, descending from the greatest to the least, in the same way with the exponents of the equation.

Note 2. If the preceding method of quadratic type indicates two imaginary roots in any part of the equation, make out the two corresponding signs of the scale of roots. That is, if $1 - f'$ be negative; also if $\frac{3}{4} - f'$ be negative, when the type has a vacant term on either side, according to the normal terms of the equation.

Note 3. Sometimes the first and third coefficient of the quadratic type, having like signs, are numerically equal to, or greater than, the middle coefficient; a pair of imaginary roots is then indicated. And to prepare for this comparison, the first and third coefficient may be, the one multiplied, and the other divided by the same assumed number,

Note 4. If a cubic type within the equation indicates two imaginary roots, mark out two corresponding signs of the scale, which are alike, leaving one sign contrary to that of the last term of the type.

The signs corresponding to a Biquadratic type may be similarly adjusted.

Example. $x^8 - 2x^5 + x^2 - 0.2 = 0$.

Scale, + + + —; that is, one negative, and one or three positive roots.

Example. $x^8 - 1.12x^7 + 2.54x^6 - 2.85x^5 - 7.37x^4 + 4.02x^3 - 0.65x^2 + 2.47x - .0387 = 0.$

By inspection with Note 3, the first three terms form one quadratic type, and the 6th 7th and 8th terms form another independent type, each indicating a pair of imaginary roots, or four in all. The remaining terms by the Rule give the scale $+ - + +$; and the corresponding real roots, found by trial, are $+ 1.65$, $- 1.25$, $+ 0.83$, $+ 0.015$. The attention of Algebraists is again invited to the double correspondence of the scale with the signs, and with the order of magnitude of the real roots.

It may be useful for future reference, to observe briefly, that the criteria of the preceding types, when placed equal to zero, coincide with the equations of condition, otherwise found, for two equal roots. And what appears more remarkable, the quantity $1 - f'$ is identical in sign with the first coefficient in the third function of Sturm's Theorem, and with the sum of the squares of the roots of the given equation, first determined by Newton; it occurs essentially in his rule of signs in a dependent order of similar elements, which might also have been taken independently.

When the ratios f, g, ... in a complete equation are all positive, the Quadratic type at once excludes each pair of consecutive ratios, where one of them exceeds 1, as pertaining to imaginary roots; and the Cubic type excludes most of the remaining pairs, where the greater triple ratio lies between 0.75 and 1. Preparatory to such considerations, we may employ the well known transformation into the equation of the squares of the roots, where all permanences of sign indicate so many imaginary roots, which will aid the complete examination. Or we may substitute $l + z$, or $- l + z$ in place of the unknown x; where l denotes the greatest limit of positive or negatve roots; and the former would render all the roots z negative, and the latter, positive. If the greatest coefficient of the equation be denoted by m, then l may be any convenient number greater than $m + 1$. In either result, the triple ratios f', g', ... will be all positive.

In the particular case of an equation having only one variation of signs, or one positive root, our method of sub-equations will give its limits. The equation is first separated into two or more portions, placed equal to zero, having the first part or highest power of x positive, and only one variation of signs. The greatest and least of the positive roots so found, will be limits, between which lies the true positive root. And their mean will be its approximate value. For example, $x^3 + 3x^2 - 6x - 8 = 0$.

Let $x_1{}^3 - 6x_1 = 0$, $3x_2{}^2 - 8 = 0$; hence $x_1 = 2.45$, $x_2 = 1.53$; between these limits is found the true positive root 2. The mean of the limits is 1.99.

For a second example, $x^4 - 3x^2 - 4x - 3 = 0$.

Let $\frac{1}{3}x_1^4 - 3x_1^2 = 0$; $\frac{1}{3}x_2^4 - 4x_2 = 0$; $\frac{1}{3}x_3^4 - 3 = 0$;

$$x_1 = 3,\ x_2 = 2.290,\ x_3 = 1.732.$$

The mean of these is 2.34; the true positive root is 2.302. In these instances, the results have not differed very widely from each other.

APPENDIX. A NEW METHOD OF APPROXIMATION.—The present contribution has extended to greater length than was first contemplated, but an important extension suggested since the commencement of writing, may still merit attention.

I. *Preliminary.* Let us premise that the double correspondence discovered in our rule of signs, will be useful in guiding the choice of the greatest, least, or special root to be found. For illustration, to commence on the right hand side of the equation, $ax^3 - bx^2 - cx - d = 0$, as in reversion of series, or in Lagrange's Theorem, would give the least root, but it is negative. To divide through first by ax^3, so as to commence on the other side, would lead to the determination of the reciprocal of the greatest root, which by inversion is the root itself. The scale of signs indicates the latter course, if the positive root is to be found.

Having chosen which root of the scale is to be computed, we next inquire which of the following approximations will be the shortest.

If the last two terms of the equation placed equal to zero, would give a small fraction of unity as the root, the approximation below for the least root will probably be rapid and easy.

If the first two terms of the equation, placed equal to zero, would give a root materially greater than unity, the process below for the greatest root will probably be the shortest. For the only positive root or the only negative root, or the two least roots or the two greatest roots, or two imaginary roots, either of the last methods can be applied, according to circumstances.

II. *The Least Root* Having transposed all the terms to one side, and divided through by the known term, let r denote the least root, and $\dots p$, q the other roots. then by the well known property of equations,

$$1 + bx + cx^2 + dx^3 + \dots = A(x-r)(x-q)(x-p)\dots = 0,$$

$$= A(x-r) \div \frac{1}{(x-q)(x-p)\dots} = 0.$$

Developing each reciprocal factor in series,

$$\frac{1}{x-q} = -\frac{1}{q} - \frac{x}{q^2} - \frac{x^2}{q^3} - \dots$$

When x represents the least root r, this series will evidently converge. Hence the divisor of $A(x-r)$ will be the product of convergent series and itself converge the more rapidly as r is relatively smaller than the other roots. The form and convergence of the divisor being known, we accordingly assume,

$$\frac{1-b'x}{1+c'x+d'x^2+e'x^3+f'x^4+\ldots\ldots} = 1+bx+cx^2+dx^3+\ldots=0.$$

When b' is thus determined, the equation $1-b'x=0$, as above shown, will give the least root. Clearing of fractions, and equating the coefficients of equal pawers af x,

$$0=b+b'+c', \qquad x=\frac{1}{b'}=\frac{-1}{b+c'}.$$

$$0=c+bc'+d',$$

$$0=d+cc'+bd'+e',$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

In successive approximations, suppose first that $c'x$ and the following terms are omitted, then $b'=-b$. Next suppose that only $d'x^2$ and the following terms are omitted; then $b'=-b+c\div b$. Next suppose that only $e'x^3$ and following terms are omitted; and so on, to determine by elimination, the value of b' more and more accurately. Let $x_1, x_2, x_3, x_4, \ldots.$ denote the corresponding values approaching the true value of x; also let $D_1, D_2, D_3, \ldots$ denote the corresponding denominators:

$$D_1=b; \qquad x_1=\frac{-1}{b}.$$

$$D_2=bD_1-c; \qquad x_2=\frac{-D_1}{D_2}=\frac{-b}{b^2-c}.$$

$$D_3=bD_2-cD_1+d; \qquad x_3=\frac{-D_2}{D_3}=\frac{-b^2+c}{b^3-2bc+d}.$$

$$D_4=bD_3-cD_2+dD_1-e; \qquad x_4=\frac{-D_3}{D_4}.$$

$$D_5=bD_4-cD_3+dD_2-eD_1+f; \quad x_5=\frac{-D_4}{D_5}. \quad \text{etc.}$$

With remarkable simplicity, the values of D are computed one from another after the manner of recurring series; and with changed sign, the ratio of two consecutive values of D gives the approximate value of x. Such is the approximation for the least root.

III. *The Greatest Root.* The preceding analysis shows we have only to commence on the other side of the equation, divided by the term containing the highest power of x. The derivation of D will be the same as before, and for x we must take the reciprocal of the former value. Thus, given the equation,

$$x^m+bx^{m-1}+cx^{m-2}+\ldots.=0.$$

$$D_1 = b;\ D_2 = bD_1 - c;\ \text{etc., as above.}$$

$$x_1 = -b,\ x_2 = -b + \frac{c}{b},\ x_3 = -b + \frac{cb - d}{b^2 - c},$$

$$x_4 = -b + \frac{c(b^2 - c) - db + e}{b(b^2 - c) - cb + d}.$$

$$x_2 = \frac{-D_2}{D_1},\ x_3 = \frac{-D_3}{D_2},\ x_4 = \frac{-D_4}{D_3}, \ldots$$

Example. Find the greatest or only positive root of the equation,

$$2x^3 - 473x^2 - 234x - 711 = 0,$$

dividing by 2 $\quad x^3 - \underset{b}{236.5}x^2 - \underset{c}{117}x - \underset{d}{355.5} = 0.$

Here $x_1 = -b = +236.5$, $x_2 = -b + c \div b = 237.00003$. The exact root found by the longer process of Horner is 237.

When a vacancy occurs after the first term, the equation may still be adapted to this method, by transposing all terms after the first, expanding the proper root of both sides, and transposing again.

When the number of vacant terms is large, as in the noted problem solved by Baily, De Morgan and others, for finding the rate of interest of an annuity, (Jones on annuities, p. 42), successive substitution is often the shortest method. The formula referred to is,

$$x = \frac{1}{p} - \frac{1}{p(1+x)^n}.\ \text{Let } x_1 = \frac{1}{p},\ x_2 = \frac{1}{p} - \frac{1}{p(1+x_1)^n},\ \text{etc.}$$

If $n = 98$ and $p = 21.924788$; $x_1 = .04561047$, $x_2 = .045034$, $x_3 = .0450017$, etc. The true root is .045.

IV. *The two Least Roots, or two Imaginary Roots.* Recurring to the analysis of article II, let another term be added to the assumed number, making it $1 - b'x - b''x^2$. Equating coefficients, we find the conditional equations the same as before, except the second, which is now $0 = c + b'' + bc' + d'$. Thus changed, the first two equations give b', b'', in terms of c', d'; which latter are found from the remaining equations, by the aid of modern determinants. Thus b', b'', being known, the two least roots, real or imaginary, are found by the quadratic,

$$1 - b'x - b''x^2 = 0;\ x = -\frac{b'}{2b''} \pm \sqrt{\left[\left(\frac{b'}{2b''}\right)^2 + \frac{1}{b''}\right]}.$$

$$b' = -b - c', \qquad b'' = -c - bc' - d'.$$

$$\begin{vmatrix} c & b & 1 & 0 & . & . \\ d & c & b & 1 & . & . \\ e & d & c & b & . & . \\ f & e & d & c & . & . \\ . & . & . & . & . & . \end{vmatrix} c' = -\begin{vmatrix} d & b & 1 & 0 & . & . \\ e & c & b & 1 & . & . \\ f & d & c & b & . & . \\ g & e & d & c & . & . \\ . & . & . & . & . & . \end{vmatrix} \Bigg\| \quad \begin{vmatrix} c & b & 1 & 0 & . & . \\ d & c & b & 1 & . & . \\ e & d & c & b & . & . \\ f & e & d & c & . & . \\ . & . & . & . & . & . \end{vmatrix} d' = +\begin{vmatrix} d & c & 1 & 0 & . & . \\ e & d & b & 1 & . & . \\ f & e & c & b & . & . \\ g & f & d & c & . & . \\ . & . & . & . & . & . \end{vmatrix}$$

$$c'_1 = 0,\ d'_1 = 0;\ c'_2 = -\frac{d}{c},\ d'_2 = 0;\ c'_3 = \frac{be - cd}{c^2 - ba},\ d'_3 = \frac{d^2 - ec}{c^2 - bd};\ldots$$

The process will be materially simplified by any deficient terms, as when $d = 0$.

To find *the two greatest roots*, we have only to divide by the term containing the highest power of x so as to bring the equation into the following form, and then apply the last process:

$$1 + b\left(\frac{1}{x}\right) + c\left(\frac{1}{x^2}\right) + \ldots\ldots = 0.$$

THE APPLICATION OF THE EXPONENTIAL POLYGON TO THE HESSIAN.

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1°. The second differential coefficient of any curve whose equation is

$$u = \psi(x, y) = 0$$

is

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 - 2\left(\frac{d^2u}{dxdy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) + \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3} \ldots\ldots (a)$$

(Todhunter's Differential Calculus, art. 180). The vanishing of (a) is the condition that the first differential coefficient is a maximum or a minimum. i. e. that the point of tangency is a point of inflection.

The condition then is

$$\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 - 2\left(\frac{d^2u}{dxdy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) + \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2 = 0 \ldots\ldots (b)$$

which is a function of x and y and is the equation of the curve called the Hessian, which may be written H $= 0$. The Hessian passes through all points of inflection of $u = 0$.

2°. At a double point of the curve $u = 0$, the first differential coefficient is indeterminate

$$\therefore \qquad \frac{dy}{dx} = -\left(\frac{du}{dx}\right) \div \left(\frac{du}{dy}\right) = \frac{0}{0}. \ldots\ldots\ldots\ldots (c)$$

Evaluate this vanishing fraction

$$\therefore \quad \frac{dy}{dx} = -\left\{\left(\frac{d^2u}{dxdy}\right)\frac{dy}{dx} + \left(\frac{d^2u}{dx^2}\right)\right\} \div \left\{\left(\frac{d^2u}{dy^2}\right)\frac{dy}{dx} + \left(\frac{d^2u}{dxdy}\right)\right\} \ldots (d)$$

(Todhunter's Differential Calculus, art. 190).